MATH 2020A Tutorial 8 26/11/2015

Topic: ∘ Stoke's thm

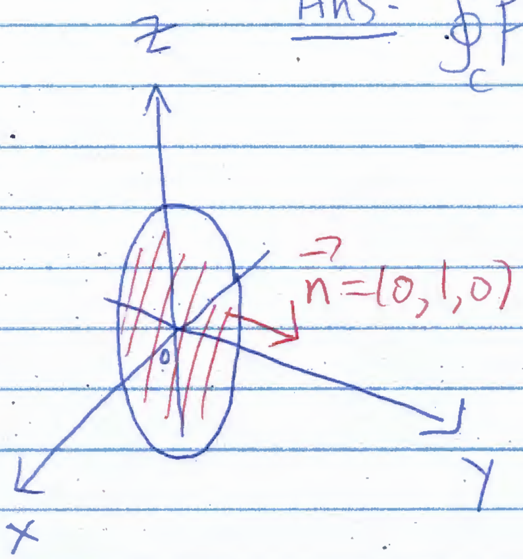
∘ Revision exercise

Stoke's thm: Let S be a piecewise smooth oriented surface having a piecewise smooth boundary C with induced orientation.
Let $\vec{F}$ be a differentiable vector field.
Then $\oint_C \vec{F}\cdot d\vec{\gamma} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\,dS$,
where $\vec{n}$ is the unit normal of S.

Example: 1) Find $\oint_C \vec{F}\cdot d\vec{\gamma}$, where

$F(x,y,z) = (x^2z+\sqrt{x^3+x+2},\ xy,\ xy+\sqrt{z^3+z^2+2})$,

C is given by $x^2+z^2=1$, $y=0$, moving anticlockwisely from the view of +ve. y-axis.



Ans: $\oint_C \vec{F}\cdot d\vec{\gamma} = \iint_D (\nabla\times\vec{F})\cdot\vec{n}\,dS$

$= \iint_D (\nabla\times\vec{F})\cdot(0,1,0)\,dS$

$= \iint_D (P_z - R_x)\,dA$

$= \iint_D (x^2-y)\,dA$

$= \iint_D x^2\,dA$

$$= \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta)\, r\,dr\,d\theta$$

$$= \frac{\pi}{4} \text{ //}$$

2) Let C be the intersection of the graphs $z = y^3$ and $x^2+y^2=3$, oriented in clockwise direction when viewed from +ve. z-axis. Find the line Integral

$$\oint_C (e^x+z)\,dx + (xy)\,dy + (ze^y)\,dz.$$

<u>Ans:</u> Let $S = \{(x,y,y^3) \mid x^2+y^2 \le 3\}$.

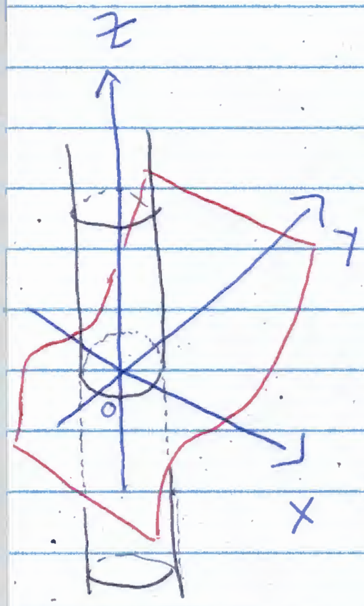
Then it can be parametrized by

$$\Sigma(u,v) = (u, v, v^3).$$

$$\Sigma_u = (1,0,0),\ \Sigma_v = (0,1,3v^2).$$

$$\Rightarrow \Sigma_u \times \Sigma_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 3v^2 \end{vmatrix} = (0,-3v^2,1)$$

By Stoke's thm,



$$\oint_C (e^x+z)\,dx + (xy)\,dy + (ze^y)\,dz$$

$$= \iint_S \nabla \times (e^x+z, xy, ze^y) \cdot (-\vec{n})\,dS$$

$$= -\iint_D (ze^y, 1, y) \cdot (0,-3v^2,1)\,dA$$

$$= -\iint_D (v-3v^2)\,dA$$

$$= -\int_0^{\sqrt{3}}\int_0^{2\pi} (r\sin\theta - 3r^2\sin^2\theta)\,d\theta\,dr$$

$$= 3^{\frac{3}{2}}\pi \text{ //}$$

Revision Exercise:

3) Let $f: \mathbb{R} \to \mathbb{R}$ be a C^1-function, γ be a piecewise C^1 curve in $\{(x,y) \mid y > 0\}$ joining the points (a,b) and (c,d).

Consider $I = \int_\gamma \frac{1}{y}[1 + y^2 f(xy)]dx + \frac{x}{y^2}[y^2 f(xy) - 1]dy$

a) Show that I is path-independent.

b) Compute I, assuming $ab = cd$.

Ans: a) It sufficies to show that $N_x = M_y$.

$$N_x = \frac{\partial}{\partial x}\left[\frac{x}{y^2}[y^2 f(xy) - 1]\right]$$

$$= \frac{\partial}{\partial x}\left[x f(xy) - \frac{x}{y^2}\right]$$

$$= f(xy) + xy f'(xy) - \frac{1}{y^2}$$

$$M_y = \frac{\partial}{\partial y}\left[\frac{1}{y}[1 + y^2 f(xy)]\right]$$

$$= \frac{\partial}{\partial y}\left[\frac{1}{y} + y f(xy)\right]$$

$$= \frac{-1}{y^2} + f(xy) + xy f'(xy)$$

$\because$ $N_x = M_y$ and $\{(x,y) \mid y > 0\}$ is simply connected

$\therefore$ I is path-independent.

b) As I is path-independent, we may choose any path γ to calculate I.

As $ab = cd$, we know that (a,b) & (c,d) lies on the curve $xy = ab = cd$

Let $\gamma(t) = (t, \frac{ab}{t})$ for $t \in [a, c]$

$\gamma(a) = (a, \frac{ab}{a}) = (a, b)$, $\gamma(c) = (c, \frac{ab}{c}) = (c, \frac{cd}{c}) = (c, d)$

So $I = \int_a^c \left(\frac{t}{ab}\left(1 + \left(\frac{ab}{t}\right)^2 f(ab)\right)(1) + \frac{t}{\left(\frac{ab}{t}\right)^2}\left(\left(\frac{ab}{t}\right)^2 f(ab) - 1\right)\left(\frac{-ab}{t^2}\right)\right) dt$

$= \int_a^c \left(\frac{t}{ab} + \frac{ab}{t} f(ab) - \frac{ab}{t} f(ab) + \frac{t}{ab} \right) dt$

$= \int_a^c \frac{2t}{ab}\, dt$

$= \frac{c}{d} - \frac{a}{b}$ //

Remark: You may find the potential function as follow:

let $\begin{cases} g_x = \frac{1}{y}(1 + y^2 f(xy)) & — ① \\ g_y = \frac{x}{y^2}(y^2 f(xy) - 1) & — ② \end{cases}$

Then $g = \int_0^x \frac{1}{y}(1 + y^2 f(ty))\, dt + C(y)$

$= \frac{x}{y} + \int_0^x f(ty)\, d(ty) + C(y)$

$= \frac{x}{y} + \int_0^{xy} f(u)\, du + C(y)$

$\Rightarrow g_y = \frac{-x}{y^2} + x f(xy) + C'(y)$

$\Rightarrow C'(y) = 0$, $C(y) = \text{constant}$.

$\Rightarrow g(x,y) = \frac{x}{y} + \int_0^{xy} f(u)\, du$ is a potential function.

$\Rightarrow I = g(c,d) - g(a,b) = \frac{c}{d} - \frac{a}{b}$ //

4) Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function.
Laplacian of a function f is defined by

$$\Delta f = \nabla \cdot (\nabla f) = f_{xx} + f_{yy} + f_{zz}.$$

A function $f: \mathbb{R}^3 \to \mathbb{R}$ is harmonic if $\Delta f = 0$.

a) Prove the Green's first identity:

$$\iint_S f \nabla g \cdot \vec{n}\, dS = \iiint_D (f \Delta g + \nabla f \cdot \nabla g)\, dV,$$

where S is the boundary of D.

Ans: $$\iint_S f \nabla g \cdot \vec{n}\, dS = \iiint_D \nabla \cdot (f \nabla g)\, dV$$

$$= \iiint_D \nabla \cdot (f g_x, f g_y, f g_z)\, dV$$

$$= \iiint_D \begin{pmatrix} f g_{xx} + f_x g_x + f g_{yy} \\ + f_y g_y + f g_{zz} + f_z g_z \end{pmatrix} dV$$

$$= \iiint_D \left[f(g_{xx} + g_{yy} + g_{zz}) + (f_x g_x + f_y g_y + f_z g_z) \right] dV$$

$$= \iiint_D (f \Delta g + \nabla f \cdot \nabla g)\, dV.$$

b) Use a) to prove that if f is harmonic on D and $f \equiv 0$ on S, then $f \equiv 0$ on D.

Ans: Put $g = f$ into a),

$$\iint_S f \nabla f \cdot \vec{n}\, ds = \iiint_D (f \Delta f + \nabla f \cdot \nabla f)\, dV$$

As $f \equiv 0$ on S and $\Delta f \equiv 0$ on D,

$$0 = \iiint_D \|\nabla f\|^2 \, dV$$

$\Rightarrow \|\nabla f\|^2 = 0$ on D

$\Rightarrow \nabla f = (0,0,0)$

$\Rightarrow f_x = f_y = f_z = 0$

$\Rightarrow f$ is a constant function on D.

As $f \equiv 0$ on S, $f \equiv 0$ on D //.